NON-COMMUTATIVE SMOOTH SPACES

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1. Motivations from Matrix Integrals

One of lessons from theoretical physics of the last decade is the following principle:

if we have integrals I_N naturally associated with $N \times N$ matrices, then the asymptotic expansion of $log(I_N)$ as $N \longrightarrow \infty$ should be

$$\sum_{g \geq 0} c_g N^{2-2g} + \text{few anomalous terms}$$

More precisely, physicists expect that a kind of non-perturbative string theory will arise automatically from matrix integrals, and $N \longrightarrow \infty$ expansion is the same as the genus expansion in string coupling constant $\lambda \sim N^{-1}$.

The origin of this principle is an old observation by t'Hooft that in the Feynman diagrammatic expansion for a system with $U(N)$-symmetry all terms are organized naturally in groups labeled by graphs drawn of oriented surfaces. Moreover, the dependence on parameter N goes only through the factor N^{2-2g}, where g is the genus of the surface. Also, many combinatorial questions concerning the enumeration of triangulations of surfaces can be analyzed using appropriate matrix models.

There are very impressive examples of the stringy behavior as $N \longrightarrow \infty$ for many models. Here are two simple cases.

A. The volume of the unitary group $U(N)$ is $const^{N^2} \prod_{j=1}^{N} \Gamma(j)$. One can check that

$$log(vol(U(N))) = +\frac{N^2 log(N)}{2} + log(\sqrt{2\pi})N + \sum_{g \geq 0} c_g N^{2-2g}$$

B. One-matrix model. Let $\Phi : \mathbf{R} \longrightarrow \mathbf{R}$ be a function increasing sufficiently fast at $\pm\infty$. Define I_N as the integral

$$\int exp(-N\, Trace(\Phi(X)))d^{N^2} X$$

taken over the space of hermitean $N \times N$ matrices. Then, as $N \longrightarrow \infty$, it has an asymptotic expansion

$$log(I_N) = -\frac{N^2 log(N)}{2} + \sum_{g \geq 0} c_g N^{2-2g}$$

For typical Φ only a term $c_0 N^2$ will appear, then in codimension 1 in the space of functions one meets an universal series related to the Airy function, etc.

In general, multi-matrix models are integrals of $exp(-N\, Trace(\Phi(X_1, \ldots, X_d)))$ where X_i are hermitean $N \times N$ matrices. In one of models in M-theory the (bosonic part of the) action Φ is $\sum [X_i, X_j]^2$. This expression is invariant under the action of the group of motions of $\mathbf{R}^d$:

$$X_i \longrightarrow \sum_j a_{ij} X_j + b_i \mathbf{1}_{N \times N}, \quad (a_{ij}) \in O(d, \mathbf{R}), \ (b_i) \in \mathbf{R}^d$$

The set of absolute minima of $Trace(\Phi)$ consists of collections of pairwise commuting hermitean matrices. The quotient of this set by the adjoint action of $U(N)$ can be naturally identified with $(\mathbf{R}^d)^N / \Sigma_N$ (configurations of N identical particles in $\mathbf{R}^d$).

2. Algebraic formulation

The manifold $M_N := \mathbf{R}^{dN^2}$ over which the multi-matrix integral is taken, can be interpreted as the set of homomorphisms of $*$-algebras: $A \longrightarrow Mat(N \times N, \mathbf{C})$ where $A = A_d$ is the free algebra $\mathbf{C}\langle x_1, \ldots, x_d \rangle$ endowed with the anti-involution $x_i^* := x_i$. This set is the set of real points in the complex affine space $\mathbf{A}^{dN^2}(\mathbf{C})$ of all homomorphisms from A to $Mat(N \times N, \mathbf{C})$.

Function Φ in the action can be considered as an element of vector space $A/[A, A]$, because traces of commutators vanish. Factor N can be written as $Trace(\mathbf{1}_A)$. In general, the expression under the exponent is linear functional $Trace \otimes Trace$ applied to an element of $A/[A, A] \otimes A/[A, A]$.

Lemma. *If ξ is a derivation of the $*$-algebra $A = A_d$, then the divergence of the induced vector field on M_N at any point $f : A \longrightarrow Mat(N \otimes N, \mathbf{C})$ is obtained by the application of the map $Trace_f \otimes Trace_f$ to an universal element $Div(\xi) \in A/[A,A] \otimes A/[A,A]$.*

It follows from this lemma that the class of measures on collections of manifolds $(M_N)_{N=1,2,\dots}$ of the form $exp((Trace \otimes Trace)(S(X_1, \dots, X_d))d^{N^2}X_1 \dots d^{N^2}X_d$ where $S \in A/[A,A] \otimes A/[A,A]$, is invariant under infinitesimal automorphisms of A.

3. Smooth algebras

Definition. *An associative algebra A over a field k is formally smooth iff for any algebra B and a nilpotent two-sided ideal I in B, and a homomorphism of algebras $A \longrightarrow B/I$, there exists a lifting $A \longrightarrow B$.*

This definition was proposed by D. Quillen and J. Cuntz. It is a copy of the standard definition of the formal smoothness in the commutative case.

Lemma. *Algebra A is formally smooth iff the bimodule $\Omega^1(A) := Ker(A \otimes A \longrightarrow A)$ (where the map is $a \otimes b \mapsto ab$) is projective. If A is formally smooth then the global cohomological dimension of the category of A-modules is ≤ 1, i.e. $Ext^{\geq 2}(E,F) = 0$ for any two A-modules E and F.*

J. Cuntz and D. Quillen found also a very explicit criterion for the formal smoothness, which is convenient for the case of finitely presented algebras.

Lemma. *If A is a finitely generated and formally smooth then for any $N \geq 1$ the scheme M_N of homomorphisms from A to the matrix algebra $Mat(N \times N, k)$ is a smooth affine scheme of finite type.*

I propose to consider smooth algebras (that is, formally smooth finitely generated algebras) as machines producing an infinite system of usual smooth schemes $(M_N)_{N=1,2,\dots}$.

There are some additional structures on manifolds M_N: for each N the group $GL(N,k)$ acts algebraically on M_N, for every pair N_1, N_2 there is a map $\oplus : M_{N_1} \times M_{N_2} \longrightarrow M_{N_1+N_2}$ equivariant with respect to $GL(N_1,k) \times GL(N_2,k) \subset GL(N_1+N_2,k)$. Manifolds M_N one can consider as a collection of N-th tensor powers of an object in a symmetric monoidal category, endowed with a compatible system of extensions of Σ_N-actions to $GL(N,k)$-actions.

Here are basic examples of smooth algebras:

a) (the main example) free finitely generated algebra $k\langle x_1, \dots, x_d \rangle$,
b) the matrix algebra $Mat(n \times n, k)$,
c) the algebra of functions on a smooth affine curve,
d) the algebra of paths in a finite oriented graph,
e) localization $A\langle x^{-1} \rangle$ where A is smooth and $x \in A$ is any element.

Smooth algebras can be considered as (dual to) "non-commutative smooth affine schemes".

Examples of non smooth algebras are algebras of functions on algebraic varieties of dimension ≥ 2, algebras of differential operators, algebras of functions on quantum groups, etc.

4. Differential geometry on smooth algebras

One can define analogues of many usual differential-geometric notions for smooth algebras. The main idea is to find a structure on algebra A producing desired structures on all representation manifolds M_N. For example, vector fields on M_N arise from derivations of A, functions arise from elements in $A/[A,A]$ (via traces), differential forms arise from elements in $\Omega^*A/[\Omega^*A, \Omega^*A]$ (Karoubi's complex). Periodic cyclic homology spaces of A for formally smooth A are (by a theorem of Cuntz and Quillen) the same as cohomology spaces of the Karoubi complex. Thus, $HP_*(A)$ maps to $H^*_{De\,Rham}(M_N)$ for any N.

One can define also the notion of a symplectic 2-form. For example, in the case $A = k\langle p \rangle / (p^2 = p)$ one has a symplectic structure on all M_N given by the formula $Trace(p\,dp\,dp)$.

P. Olver discovered generalizations of integrable systems (such as KdV) to the free noncommutative case. Passing to spaces of matrix representations one get a sequence of ordinary integrable systems.

5. Completion near the commutative quotient

Recently M. Kapranov developed a theory of formal noncommutative thickenings of usual manifolds. The structure of a formal noncommutative thickening on a usual manifold X is given by a sheaf of algebras

$\mathcal{O}_{noncomm}$ mapping to $\mathcal{O}_X$. In the case $X = \mathbf{A}^d$ the algebra of global sections $\mathcal{O}_{noncomm}(X)$ is the completion of the free algebra $k\langle x_1, \ldots, x_d\rangle$ by the filtration by the total number of commutators. If X is itself formal, say $FormSpec(k[[x_1, \ldots, x_d]])$, then $\mathcal{O}_{noncomm}(X)$ is the algebra of formal noncommutative power series $k\langle\langle x_1, \ldots, x_d\rangle\rangle$.

It is easy to see if A is a smooth algebra then M_1 has a canonical formal noncommutative thickening.

The importance of formal noncommutative thickenings is explained by the following observation made by Kapranov:

if C is a triangulated category with finite-dimensional Hom-s, and $\alpha : E \longrightarrow F$ is a morphism, such that the tangent complex for the deformation theory of it with fixed object E has vanishing H^{-1} and H^1, then the germ of the formal moduli space of α with fixed source E, is smooth and has a canonical noncommutative thickening.

One can show that to have a noncommutative thickening of an algebraic manifold X/k is exactly the same as to have a system of formal manifolds (M_N) endowed with $GL(N)$-actions and addition maps, such that each (M_N) is a thickening in the usual commutative sense of the scheme of pairs (E, iso) where E is a coherent sheaf with finite support on X, of length N, and iso is an isomorphism between vector spaces $\Gamma(X, E)$ and k^N. This scheme is similar to the set of minima in M-theory (see sect. 1).

6. Compactifications

It is very natural to try to define "compact" smooth noncommutative manifolds as certain machines producing as "representation spaces" sequences of usual smooth projective manifolds M_N (endowed with $GL(N)$-actions and addition maps). With A. Rosenberg we proposed a general definition of a noncommutative scheme. It is a beginning of a large project, and I am not going to describe it here at all. I only mention that in our framework there are coherent sheaves and cohomology as in the usual commutative algebraic geometry.

We have now a collection of examples which are formally smooth, and produce always smooth projective varieties. For example, our "universal grassmanian of subspaces in 1-dimensional space" has as the set of $Mat(N \times N, k)$-points the disjoint union of all grassmanians

$$\bigsqcup_{0 \le n \le N} Gr(n, N)(k)$$

We can make products, blow-ups, and other constructions. Moduli spaces (stacks) of coherent sheaves on compact smooth non-commutative spaces are smooth, and also are abelianizations of certain smooth noncommutative spaces. In all examples the category of coherent sheaves has global cohomological dimension 1, and all Hom-spaces and Ext^1-spaces are finite-dimensional. Also, the categories of coherent sheaves are complete in certain sense. Moreover, one can define moduli spaces of such categories, and by purely formal reasons the moduli space is a smooth stack of finite type. Thus, we have generalizations of the moduli stack of algebraic curves. One of our moduli stacks parametrizes collections of disjoint projective subspaces in a projective space.

Morally, all noncommutative smooth spaces are "curves". One can try to define generalizations of Gromov-Witten invariants considering spaces of maps between noncommutative compact smooth spaces. Taking the tensor product of two smooth non-commutative spaces we obtain a singular space with the category of coherent sheaves of cohomological dimension 2 ("non-commutative algebraic surfaces"). For such categories virtual fundamental classes of moduli spaces of objects can be defined (at least locally). One can hope to have generalized Donaldson invariants for such two-dimensional spaces.

7. Further topics

I hope that the language and the new intuition of smooth algebras will be useful someday for more familiar examples, such as deformations of commutative algebras. The reason is that many usual algebras admit resolutions by finitely generated differential graded smooth algebras. Thus, noncommutative differential graded supermanifolds can serve as replacements of usual algebras.

Derived categories of abelian categories of global cohomological dimension ≤ 1 form a natural arena for constructions of Hall-Ringel algebras. It is known that all Kac-Moody Lie algebras of finite rank appear (as Lie subalgebras) in this way. It seems plausible that Borcherds' infinite rank algebras with Monstrous symmetry can be realized inside Hall-Ringel algebras for some small smooth noncommutative spaces.

Also, one can play a purely mathematical game imitating "physics" of matrix models. In this gane integrals are replaced by numbers of points over finite fields. Instead of taking the asymptotic expansion at $N \longrightarrow \infty$ one can make generating functions. Here one get a feeling that the modular type of behavior is ubiquitous. Here are few examples:

Number of points in $M_N(\mathbf{F}_q)$ for the free algebra $\mathbf{Z}\langle x_1, \ldots, x_d\rangle$ is q^{dN^2}. If we now consider q as a formal parameter, the sum over all $N \in \mathbf{Z}$ will be a modular form. Analogously, counting points in $GL(N, \mathbf{F}_q)$ (i.e. homomorphisms $\mathbf{Z}[t, t^{-1}] \longrightarrow Mat(N \times N, \mathbf{F}_q)$), we get

$$\sum_N \left(q^{N(N-1)/2} \prod_{j=1}^{N} (q^j - 1) \right) z^N$$

which seems to be a quasi-modular form.

If we consider q as an actual number, then the leading term in the formula for numbers of $Mat(N \times N, \mathbf{F}_q)$-points for the algebra $\mathbf{Z}\langle p\rangle/(p^2 = p)$ is (for even N)

$$q^{3N^2/4} \left(\sum_{n \in \mathbf{Z}} q^{-3n^2} \right) \prod_{j=1}^{\infty} (1 - q^{-j})$$

Analogous results hold in other examples, including the compact ones. Physicists told me that this modular behavior does not look totally surprising for them, because the size of matrices plays the role of a variable dual to certain compactification length in an extra dimension, whatever it means.

Sometimes one can calculate an integral over a manifold reducing it to a "topological integral" (an expression in characteristic classes of some vector bundles) over a compactification of the original space. Several matrix integrals calculated recently in physical literature, have rational values. Thus, they have a chance to be calculable topologically. I hope that noncommutative compactifications constructed by Rosenberg and myself, can be useful in some matrix models. The following question arises naturally: how cohomology of manifolds M_N behave as N grow? For example, what can one say about the generating function of the Euler characteristics of spaces M_N?